THE CONSTRUCTION OF THE XWINDOW FUNCTION

(used in time - series and signal processing)

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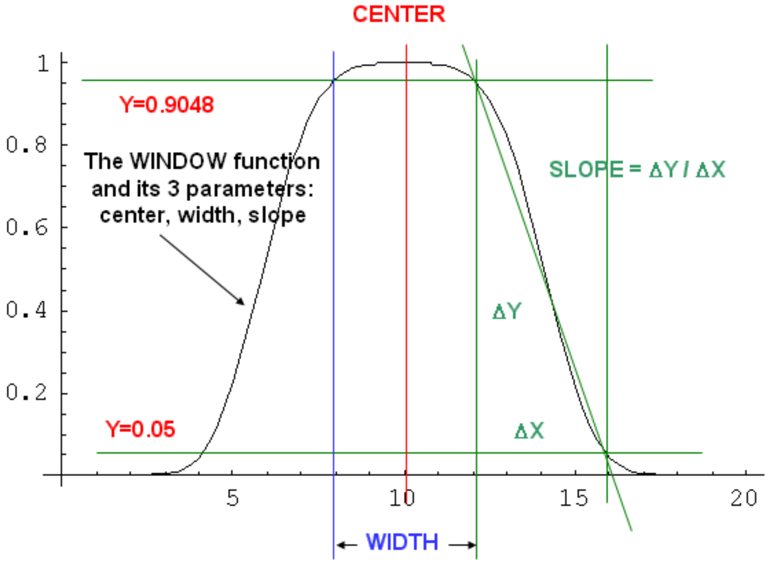
Using Mathematica 12.0

XWindow[x_real], the eXtended Window function,
is a Heaviside, Impulse or Rectangle - like function,
with four characteristics (see next figure) :

1. It is continuous from minus to plus infinity;
2. It' s Center is an input parameter
3. It' s Slope is an input parameter
4. It' s Width is an input parameter

Start by reading the output of this notebook.



In[91]:=

```
Clear[f, x, c, s, w, center, width, slope];

(* We start with function f[x]: *)
```

$$f[x\_] := \mathrm{Exp}\left[-\left(\frac{x-c}{s}\right)^{w}\right];$$

```
(* Note that when w=2, f is the Gauss (or Bell) function *)

    c = 10; s = 4.1; w = 10;
    Print["The Bell Curve (blue) and Window[x] (red) : ", f[x]];
```

$$\mathrm{Plot}\left[\left\{f[x], \mathrm{Exp}\left[-\left(\frac{x-c}{s}\right)^{2}\right]\right\}, \{x, 0, 20\}, \mathrm{AxesOrigin} \to \{0, 0\},\right.$$

```
    PlotRange → All, PlotStyle → {{RGBColor[1, 0, 0]}, {RGBColor[0, 0, 1]}}];

(*
Now we change the variables from {c,w,s} to {center,width,slope},
    according to the figure shown above.
The value of x at the center of the curve is c = center.
When f[x]=0.9048, x = x1 > c and width = 2 * ( x1 - c ):
    Solve[f==0.9048,x] (* to find X1 *);
```

$x1 = c + \frac{s}{10^{\frac{1}{w}}}$

width $= \frac{2\ s}{10^{\frac{1}{w}}}$; isolating s on the left:

$s = \frac{width}{2}\ 10^{\frac{1}{w}}$;

```
Now verify this result.
*)

Clear[w, c];
c = center;
```

$s = \left(\frac{width}{2}\right)\left(10^{\frac{1}{w}}\right)$;

```
center = 10; width = 4; w = 4;
Print["Center = ", center];
Print["Plot f[x] for width = ", width]
Print["Function f[x] being plotted: ", f[x]];
Plot[f[x], {x, 0, 20}, AxesOrigin → {0, 0}, PlotRange → All]

(*
```

Now we search for the value of the slope = $\frac{\Delta y}{\Delta x}$; (see figure above);

$x1 = c + \frac{s}{10^{\frac{1}{w}}}$; $f = e^{-\left(\frac{x-c}{s}\right)^{w}}$; (* already found *)

Solve[f==0.05,x] (* find x2 *);

$x2 = c + s*3^{\left(\frac{1}{w}\right)}$;

$\Delta x = x2 - x1$; $\Delta y = 0.9$;

slope= $\frac{\Delta y}{\Delta x}$;

slope= $\frac{0.9}{\left(3^{\frac{1}{w}} - 10^{-\frac{1}{w}}\right)\ s}$; isolating s on the left:

$s = \frac{0.9}{\left(3^{\frac{1}{w}} - 10^{-\frac{1}{w}}\right)\ slope}$;

But

$s = \frac{width}{2}\ 10^{\frac{1}{w}}$; (* See above *)

Solving this system of 2 eqs & 2 unknowns:

$w = \frac{Log[30]}{Log\left[1 + \frac{1.8}{slope * width}\right]}$

Once defined the values of {w,s,c}, we get a pre-definition of Window[x]:

$k = 1 + \frac{1.8}{slope * width}$;

preWindow[x] = $Exp\left[-\left(\frac{2\ k^{-\frac{Log[10]}{Log[30]}}\ (x-c)}{width}\right)^{\frac{Log[30]}{Log[k]}}\right]$;

Because Window[x] is an even function, we must use Abs[x-c] instead of (x-c), or (which is the same) making $(x-c) \to \left((x-c)^2\right)^{\frac{1}{2}}$, a continuous function.

With above considerations, Window[x] finally becomes:

$$\text{Window}[x] = \text{Exp}\left[-\left(k^{-\text{Log}[10]}\left(\left(\frac{x-\text{center}}{\text{width}/2}\right)^2\right)^{\frac{\text{Log}[30]}{2}}\right)^{\frac{1}{\text{Log}[k]}}\right]$$

Note that Window[x] is a function of the type u^{x^z}.

To avoid division by zero,

$\text{Log}[k] > 0,\ k > 1,\ 0 < \text{width} < \infty,\ 0 < \text{slope} < \infty;$

If Abs[x-c] is too big, underflows may result during computation.
To avoid this, we define the range of Window[x]:

$0 \le \text{Window}[x] < \text{min}$

where

$\text{min} = 2.225\ 10^{-308}$ for a double-precision floating-number in C++ (float.h);

$\text{min} = 5.208\ 10^{-646456888}$ in Mathematica (calling $MinNumber);

or

$\text{min} = \text{Exp}[-708.4]$ (C++)

$\text{min} = \text{Exp}[-1.48\ 10^9]$ (Mathematica)

To find the maximum of Abs[x-c], we equate

Window[xuf] == min (* xuf → min x that causes underflow *)

to obtain the condition to avoid underflow:

$$\text{Abs}[\text{xuf}-c] < \left(\frac{\text{width}}{2}\right) * k^{\left(\frac{\text{Log}[\text{maxExpo}] + \text{Log}[10]}{\text{Log}[30]}\right)}$$

where

$\text{maxExpo} = 708.4$ (for C++)

$\text{maxExpo} = 1.48\ 10^9$ (for Mathematica)

For C++

$\text{Abs}[x-c] < \frac{\text{width}}{2}\ k^{2.6066}$

For Mathematica

$\text{Abs}[x-c] < \frac{\text{width}}{2}\ k^{6.88}$

To reduce the time used to compute Window[x], we may observe that

if x >> c, Window[x] → zero.

Then, if

Window[xfar > c] == 0.001

the value of xfar is

$\text{Abs}[\text{xfar}-c] = \frac{\text{width}}{2}\ k^{1.245}$

Also, when window[x05]==0.05 → $(\text{x05}-\text{center}) = \frac{\text{width}}{2} k$

We note that:

$\text{Abs}[\text{x05}-\text{center}] = \frac{\text{width}}{2} k < \text{Abs}[\text{xfar}-c] = \frac{\text{width}}{2} k^{1.245} <$

$< \text{Abs}[\text{xuf1}-c] = \frac{\text{width}}{2} k^{2.6066} << \text{Abs}[\text{xuf2}-c] = \frac{\text{width}}{2} k^{6.88}$

```
In words,
    (1) xfar is less then the underflow limit xuf
    (2) xfar is near x05

We may resume the above considerations with the command:

    If[ Abs[x-c] > (width/2) k^1.245,
        Return[0],
        Return[Window[x];
    ]

A complete definition of Windows[x] (in red) is shown in the next Module[].
   *)

(* Mathematical expression of window[x]:
```

$$\text{window}[x\_,center\_,width\_,slope\_] := \text{Exp}\left[-\left(\frac{\left(\left(\frac{x-center}{width/2}\right)^2\right)^{\frac{Log[30]}{2}}}{k^{\,Log[10]}}\right)^{\frac{1}{Log[k]}}\right];$$

```
*)

    Clear[slope, width, center, k, window];
    k := 1 + 1.8/(slope * width);

    window[x_, center_, width_, slope_] :=
        Module[{},
            If[ width ≤ 0, Print["Width must be > 0: ", width];
    Return["ERROR"]];
            If[ slope ≤ 0, Print["Slope must be > 0: ", slope];
    Return["ERROR"]];
            If[Abs[x - center] > (width/2) k^1.245, Return[0]];
```

$$\text{Return}\left[\text{Exp}\left[-\left(k^{-Log[10]}\left(\left(\frac{x-center}{width/2}\right)^2\right)^{\frac{Log[30]}{2}}\right)^{\frac{1}{Log[k]}}\right]\right]$$

```
            ];

    "center=0;width=6;slope=2;"
    center = 0; width = 6; slope = 2;

    Print["Plotting window[x] for slope = ", slope];
    Print[window[x, center, width, slope]];

Plot[window[x, center, width, slope], {x, -10, 10}, AxesOrigin → {0, 0}, PlotRange → All];

    (* Now plot window[x] when center, width and slope change *)

Print[];
Print["Window[x] when its center is changed (below)"];
Clear[x, center];
"width=10; slope=0.5;"
width = 10; slope = 0.5;
```

```
Print[window[x, center, width, slope] // TraditionalForm];
Plot3D[window[x, center, width, slope], {x, -40, 40}, {center, -10, 10},
 PlotRange → All, AxesLabel → {"X value", "Center Point", "Window[x]"},
 Mesh → False, FaceGrids → All, PlotPoints → 100]

Print[];
Clear[x, width];
Print["Window[x] when its width is changed (below)"];
Clear[width];
"center=0;slope=0.06;"
center = 0; slope = 0.06;
Print[window[x, center, width, slope] // TraditionalForm];
Plot3D[window[x, center, width, slope], {x, -40, 40}, {width, 2, 40},
 PlotRange → All, AxesLabel → {"X value", "Width", "Window[x]"},
 Mesh → False, FaceGrids → All, PlotPoints → 100]

Print[];
Clear[x, slope];
Print["Window[x] when its slope is changed (below)"]
Clear[slope];
"center=0;width=50;"
center = 0; width = 50;
Print[window[x, center = 0, width, slope] // TraditionalForm];
Plot3D[window[x, center = 0, width, slope], {x, -600, 600},
 {slope, 0.001, .03}, PlotRange → All, AxesLabel → {"X value", "Slope", "Window[x]"},
 Mesh → False, FaceGrids → All, (*ColorOutput→GrayLevel,*)PlotPoints → 200]

(* END OF JOB *)
```

Null

The Bell Curve (blue) and Window[x] (red) : $e^{-7.45009\times10^{-7}(-10+x)^{10}}$

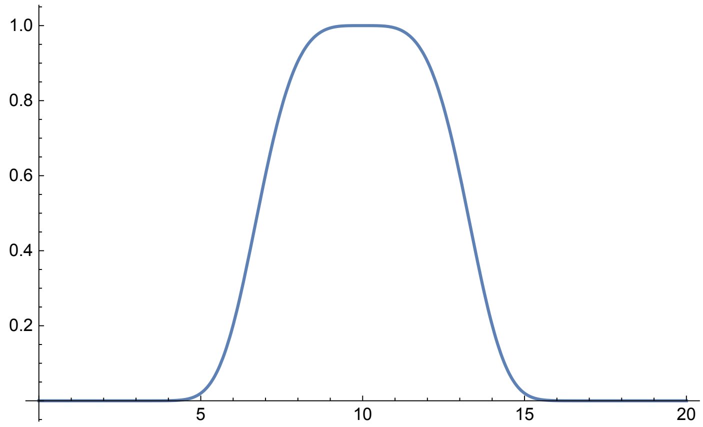
... General: Exp[−7447.05] is too small to represent as a normalized machine number; precision may be lost.

Center = 10

Plot f[x] for width = 4

Function f[x] being plotted: $e^{-\frac{1}{160}(-10+x)^4}$

Out[103]=



Out[107]= center=0;width=6;slope=2;

Plotting window[x] for slope = 2

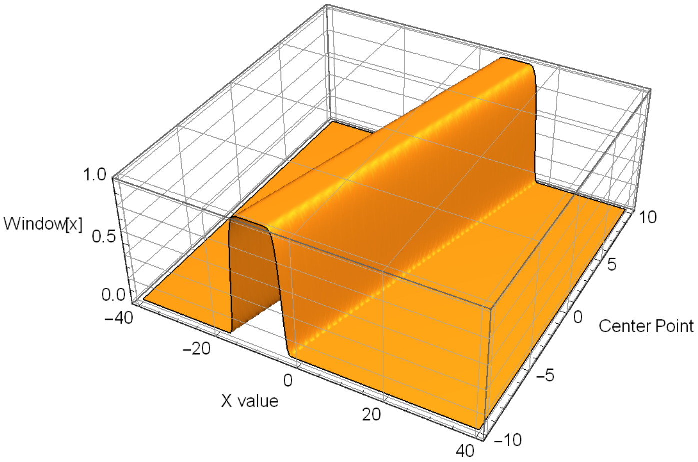
$$e^{-2.44875\times10^{-13}\left((x^2)^{\frac{\text{Log}[30]}{2}}\right)^{7.15502}}$$

Window[x] when its center is changed (below)

Out[115]= width=10; slope=0.5;

$$e^{-1.85543\times10^{-9}\left(((x-\text{center})^2)^{\frac{\log(30)}{2}}\right)^{3.25219}}$$

Out[118]=

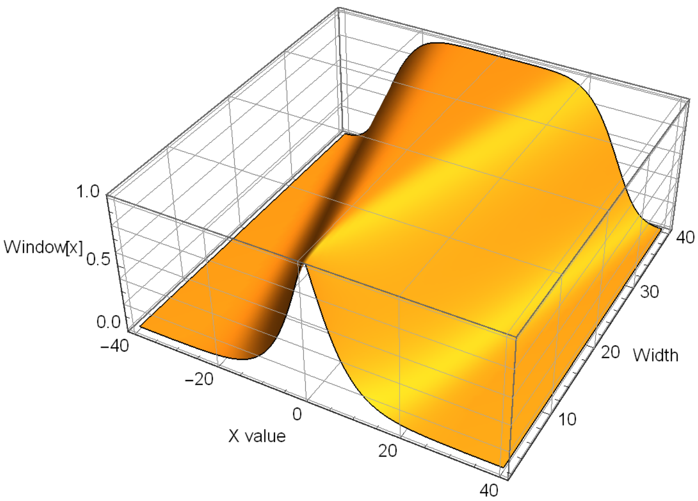


Window[x] when its width is changed (below)

Out[123]= center=0;slope=0.06;

$$\exp\left(-2^{\frac{\log(30)}{\log\left(\frac{30.}{\text{width}}+1\right)}}\left(\left(\frac{30.}{\text{width}}+1\right)^{-\log(10)}\left(\frac{x^2}{\text{width}^2}\right)^{\frac{\log(30)}{2}}\right)^{\frac{1}{\log\left(\frac{30.}{\text{width}}+1\right)}}\right)$$

Out[126]=



Window[x] when its slope is changed (below)

Out[131]= center=0;width=50;

$$\exp\left(-25^{-\frac{\log(30)}{\log\left(\frac{0.036}{\text{slope}}+1\right)}}\left(\left(\frac{0.036}{\text{slope}}+1\right)^{-\log(10)}\left(x^2\right)^{\frac{\log(30)}{2}}\right)^{\frac{1}{\log\left(\frac{0.036}{\text{slope}}+1\right)}}\right)$$

Out[134]=